

Deriving the cubic formula

Consider the cubic equation in the form $a_0 + a_1z + a_2z^2 + a_3z^3 = 0$, where $a_3 \neq 0$.

1. Divide both sides of the equation by a_3 :

$$\frac{a_0}{a_3} + \frac{a_1}{a_3}z + \frac{a_2}{a_3}z^2 + z^3 = 0$$

2. Substitute $z \leftarrow x - \frac{a_2}{3a_3}$ and expand:

$$\begin{aligned} \frac{a_0}{a_3} + \frac{a_1}{a_3}\left(x - \frac{a_2}{3a_3}\right) + \frac{a_2}{a_3}\left(x - \frac{a_2}{3a_3}\right)^2 + \left(x - \frac{a_2}{3a_3}\right)^3 &= 0 \\ \frac{a_0}{a_3} + \left(\frac{a_1}{a_3}x - \frac{a_1a_2}{3a_3^2}\right) + \left(\frac{a_2}{a_3}x^2 - \frac{2a_2^2}{3a_3^2}x + \frac{a_2^3}{9a_3^3}\right) + \left(x^3 - \frac{a_2}{a_3}x^2 + \frac{a_2^2}{3a_3^2}x - \frac{a_2^3}{27a_3^3}\right) &= 0 \\ \left(\frac{a_0}{a_3} - \frac{a_1a_2}{3a_3^2} + \frac{2a_2^3}{27a_3^3}\right) + \left(\frac{a_1}{a_3} - \frac{a_2^2}{3a_3^2}\right)x + x^3 &= 0 \end{aligned}$$

3. For simplicity, define

$$\begin{aligned} b_0 &\equiv \frac{a_0}{a_3} - \frac{a_1a_2}{3a_3^2} + \frac{2a_2^3}{27a_3^3}, \\ b_1 &\equiv \frac{a_1}{a_3} - \frac{a_2^2}{3a_3^2}, \end{aligned}$$

reducing the equation to $b_0 + b_1x + x^3 = 0$.

4. Substitute $x \leftarrow w - \frac{b_1}{3w}$ and expand:

$$\begin{aligned} b_0 + b_1\left(w - \frac{b_1}{3w}\right) + \left(w - \frac{b_1}{3w}\right)^3 &= 0 \\ b_0 + \left(b_1w - \frac{b_1^2}{3w}\right) + \left(w^3 - b_1w + \frac{b_1^2}{3w} - \frac{b_1^3}{27w^3}\right) &= 0 \\ -\frac{b_1^3}{27w^3} + b_0 + w^3 &= 0 \end{aligned}$$

5. Multiply through by w^3 to obtain a quadratic in w^3 :

$$\left(-\frac{b_1}{3}\right)^3 + b_0w^3 + (w^3)^2 = 0$$

6. Solve for w^3 using the quadratic formula:

$$w^3 = -\frac{b_0}{2} - \sqrt{\left(\frac{b_0}{2}\right)^2 + \left(\frac{b_1}{3}\right)^3} \quad \left(\text{alternatively, } w^3 = -\frac{b_0}{2} + \sqrt{\left(\frac{b_0}{2}\right)^2 + \left(\frac{b_1}{3}\right)^3}\right)$$

Every non-zero complex number has three distinct cube roots. Choosing any one of these for w and back-substituting yields all three solutions to the cubic equation:

$$z = \sqrt[3]{-\frac{b_0}{2} - \sqrt{\left(\frac{b_0}{2}\right)^2 + \left(\frac{b_1}{3}\right)^3}} - \frac{b_1}{3\sqrt[3]{-\frac{b_0}{2} - \sqrt{\left(\frac{b_0}{2}\right)^2 + \left(\frac{b_1}{3}\right)^3}}} - \frac{a_2}{3a_3}.$$

This formula only fails if both choices for w^3 are zero (i.e., $b_0 = b_1 = 0$). In this case, the cubic has a triple root at $z = -\frac{a_2}{3a_3}$.

Alternative trigonometric solution

Starting from the reduced cubic equation, it is possible to derive a trigonometric cubic formula.

1. Substitute $x \leftarrow \frac{2\sqrt{-b_1}}{\sqrt{3}} \cos(w)$ into the equation:

$$\begin{aligned} b_0 + b_1 \left(\frac{2\sqrt{-b_1}}{\sqrt{3}} \cos(w) \right) + \left(\frac{2\sqrt{-b_1}}{\sqrt{3}} \cos(w) \right)^3 &= 0 \\ b_0 - \frac{2(-b_1)^{3/2}}{\sqrt{3}} \cos(w) + \frac{8(-b_1)^{3/2}}{3\sqrt{3}} (\cos(w))^3 &= 0 \end{aligned}$$

2. Multiply through by $\frac{3\sqrt{3}}{2(-b_1)^{3/2}}$:

$$\frac{3\sqrt{3}b_0}{2(-b_1)^{3/2}} - 3\cos(w) + 4(\cos(w))^3 = 0$$

Using the trigonometric identity $4(\cos(w))^3 - 3\cos(w) = \cos(3w)$, this simplifies to

$$\frac{3\sqrt{3}b_0}{2(-b_1)^{3/2}} + \cos(3w) = 0.$$

3. Solve for w using the inverse cosine function:

$$w = \frac{1}{3} \arccos \left(-\frac{3\sqrt{3}b_0}{2(-b_1)^{3/2}} \right)$$

Due to the periodicity of cosine, there remains three distinct solutions for x , which after back-substituting $z = x - \frac{a_2}{3a_3}$ gives the three roots of the original cubic equation:

$$z = \frac{2\sqrt{-b_1}}{\sqrt{3}} \cos \left(\frac{1}{3} \arccos \left(-\frac{3\sqrt{3}b_0}{2(-b_1)^{3/2}} \right) + \frac{2\pi n}{3} \right) - \frac{a_2}{3a_3}, \text{ for } n = 0, 1, 2.$$

This approach fails if $b_1 = 0$, reducing the equation to $b_0 + x^3 = 0$, which is trivially solvable.